

MAS212 Linear Algebra I Exercises 9 (with solutions)

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Assessed submission deadline 5pm on Friday 8th December 2006 in the blue box on the ground floor.

Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together.

Marks out of 100 are shown in square brackets after each question number.

1. [10] Use elementary row operations to reduce the matrix $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 5 & 4 & 6 \\ 5 & 7 & 6 & 8 \end{bmatrix}$ to echelon form.

Then use the echelon form of A to solve the following problem: (a) use rank arguments to find the

nature of the solution of the matrix equation $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 5 & 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$ and (b) find the solution

explicitly.

▼ **Solution**

Row-reducing A gives sucessively

$$R_2 \rightarrow R_2 - 3 R_1$$

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -4 & -2 & -6 \\ 5 & 7 & 6 & 8 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 5 R_1$$

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -4 & -2 & -6 \\ 0 & -8 & -4 & -12 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2 R_2$$

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -4 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The augmented matrix of the equation is A , but the echelon form of A is not determined by the last column. Hence, the echelon form of the coefficient matrix in the equation is the same as that of A with the last column removed. In both cases, there are two nonzero rows, so the rank of both the coefficient matrix and the augmented matrix (A) is the same, namely 2. Hence the equation has a 1-dimensional (affine) solution space, i.e. the solution involves one parameter.

(b) The scalar equations in echelon form are

$$x + 3y + 2z = 4$$

$$-4y - 2z = -6$$

$$0 = 0$$

Regarding z as a parameter (since there are only 2 equations and 3 unknowns), the solution (by back-substituting) is

$$y = -\frac{1}{2}z + \frac{3}{2}$$

$$x = -\frac{1}{2}z - \frac{1}{2}$$

► Maple Solution

2. [30] Use reduction to echelon form to compute each of the following determinants:

$$(a) \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix}; \quad (b) \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{vmatrix}; \quad (c) \begin{vmatrix} 0 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 0 \end{vmatrix}.$$

▼ Solution

(a) Row-reduce to echelon form keeping track of the number of row interchanges:

$$R_1 \leftrightarrow R_3$$

1 interchange

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \frac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 0 & 0 & \frac{7}{4} \end{bmatrix}$$

By multiplying together the diagonal elements of the triangular matrix and allowing for the sign change caused by the odd number of row interchanges, the determinant is $-(1)(-4)\left(\frac{7}{4}\right) = 7$.

(b) Row-reduce to echelon form keeping track of the number of row interchanges:

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -7 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2 R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

No interchanges were required. So by multiplying the diagonal elements of the triangular matrix, the determinant is $(1) (-1) (-3) = 3$.

(c) Row-reduce to echelon form keeping track of the number of row interchanges:

$$R_2 \leftrightarrow R_1$$

$$\begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 3 \\ 3 & 2 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{3}{2} R_1$$

$$\begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & -\frac{5}{2} & -3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \frac{5}{4} R_2$$

$$\begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}$$

By multiplying together the diagonal elements of the triangular matrix and allowing for the sign change caused by the odd number of row interchanges, the determinant is $-(2) (2) \left(\frac{3}{4}\right) = -3$.

► Maple Solution

3. [30] For each of the following matrices, use reduction to row canonical form to compute the inverse if it exists, in which case also check that the result is correct, or to show that the inverse does not exist:

$$(a) \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}; \quad (b) \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}; \quad (c) \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

▼ Solution

(a) Adjoin the identity matrix to A to give

$$\begin{bmatrix} 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and reduce to row canonical form [the following order of operations is not necessary]

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow -R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow -\frac{1}{2} R_3$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_2 + R_3 \\ R_1 &\rightarrow R_1 - R_3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Hence A^{-1} exists and is given by

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Check:

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Adjoin the identity matrix to A to give

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 5 & 6 & 7 & 0 & 0 & 1 \end{bmatrix}$$

and reduce to row canonical form [the following order of operations is not necessary]

$$\begin{array}{l} R_2 \rightarrow R_2 - 3 R_1 \\ R_3 \rightarrow R_3 - 5 R_1 \end{array} \quad \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -4 & -3 & 1 & 0 \\ 0 & -4 & -8 & -5 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow -\frac{1}{2} R_2 \quad \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -4 & -8 & -5 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 4 R_2 \quad \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Cannot reduce A to the identity (since it has rank 2), so A^{-1} does not exist.

(c) Adjoin the identity matrix to A to give

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and reduce to row canonical form [the following order of operations is not necessary]

$$\begin{array}{l} R_2 \rightarrow R_2 - 2 R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \quad \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -6 & -2 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow -\frac{1}{5} R_2 \quad \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{6}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R_3 &\rightarrow R_3 + 2 R_2 \\ R_1 &\rightarrow R_1 - 2 R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{6}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} & 1 \end{bmatrix}$$

$$R_3 \rightarrow \frac{5}{2} R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{6}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -1 & \frac{5}{2} \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_2 - \frac{6}{5} R_3 \\ R_1 &\rightarrow R_1 - \frac{3}{5} R_3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 1 & -\frac{3}{2} \\ 0 & 1 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -\frac{1}{2} & -1 & \frac{5}{2} \end{bmatrix}$$

Hence A^{-1} exists and is given by

$$\begin{bmatrix} \frac{1}{2} & 1 & -\frac{3}{2} \\ 1 & 1 & -3 \\ -\frac{1}{2} & -1 & \frac{5}{2} \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 1 & -\frac{3}{2} \\ 1 & 1 & -3 \\ -\frac{1}{2} & -1 & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

► Maple Solution

4. [10] Let A and B be two $m \times n$ matrices over a field $\mathbb{K}$ and denote by $A \sim B$ the fact that there exist nonsingular matrices P and Q over the field $\mathbb{K}$ such that $B = P A Q$, where P and Q depend on A and B . Prove that $\sim$ is an equivalence relation, i.e. it is reflexive ($A \sim A$), symmetric ($A \sim B \Rightarrow B \sim A$) and transitive ($A \sim B$ and $B \sim C \Rightarrow A \sim C$).

▼ Solution

reflexivity ($A \sim A$) : $P = Q = I \Rightarrow A = P A Q$.

symmetry ($A \sim B \Rightarrow B \sim A$) : $B = P A Q \Rightarrow A = P^{-1} B Q^{-1}$.

transitivity ($A \sim B$ and $B \sim C \Rightarrow A \sim C$) : $B = P A Q$ and $C = R B S$, where R and S are nonsingular matrices $\Rightarrow C = (R P) A (Q S)$, where $R P$ and $Q S$ are nonsingular matrices.

5. [20] Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 4 & 2 & 5 & 1 \end{bmatrix}$. Use a combination of elementary row and column operations to find

nonsingular matrices P and Q such that $PAQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution

(a) Adjoin the 3×3 identity matrix to A to give

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 & 0 & 1 & 0 \\ 4 & 2 & 5 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and reduce to row canonical form:

$$\begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array} \quad \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -6 & -7 & 1 & -3 & 1 & 0 \\ 0 & -6 & -7 & 1 & -4 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow -\frac{1}{6}R_2 \quad \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{6} & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & -6 & -7 & 1 & -4 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_3 \rightarrow R_3 + 6R_2 \\ R_1 \rightarrow R_1 - 2R_2 \end{array} \quad \begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{7}{6} & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

The first 4 columns and the last 3 columns give respectively matrices B and P such that $B = PA$ where

$$B := \begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{7}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P := \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & -\frac{1}{6} & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Check:

$$PA = \begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{7}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now either column-reduce, or transpose and row-reduce B with a 4×4 identity matrix adjoined; I

will do the latter (as in the lecture notes):

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \frac{2}{3} & \frac{7}{6} & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R_3 &\rightarrow R_3 - \frac{2}{3} R_1 \\ R_4 &\rightarrow R_4 - \frac{1}{3} R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{7}{6} & 0 & -\frac{2}{3} & 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 0 & -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R_3 &\rightarrow R_3 - \frac{7}{6} R_2 \\ R_4 &\rightarrow R_4 + \frac{1}{6} R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{3} & -\frac{7}{6} & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & 0 & 1 \end{bmatrix}$$

The first 3 columns and the last 4 columns give respectively matrices B^T and Q^T such that $B = P A Q$ where

$$B := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q := \begin{bmatrix} 1 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{7}{6} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check:

$$P.A.Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

► **Maple Solution**